Discrete Mathematics 310 (2010) 714–719

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Posets associated with subspaces in a d-bounded distance-regular graph

Jun Guo [a], Kaishun Wang [b,*]

[a] Math. and Inf. College, Langfang Teachers' College, Langfang 065000, China

[b] Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing 100875, China

ARTICLE INFO

Article history:
Received 10 February 2009
Received in revised form 18 June 2009
Accepted 31 August 2009
Available online 17 September 2009

Keywords:
Distance-regular graph
Subspace
Poset
Characteristic polynomial

ABSTRACT

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 3$. A regular strongly closed subgraph of Γ is said to be a subspace of Γ. For $x \in X$, let $P(x)$ be the set of all subspaces of Γ containing x. For each $i = 1, 2, \ldots, d-1$, let Δ_0 be a fixed subspace with diameter $d - i$ in $P(x)$, and let

$$\mathscr{L}(d, i) = \{\Delta \in P(x) \mid \Delta + \Delta_0 = \Gamma, d(\Delta) = d(\Delta \cap \Delta_0) + i\} \cup \{\emptyset\}.$$

If we define the partial order on $\mathscr{L}(d, i)$ by ordinary inclusion (resp. reverse inclusion), then $\mathscr{L}(d, i)$ is a finite poset, denoted by $\mathscr{L}_O(d, i)$ (resp. $\mathscr{L}_R(d, i)$). In the present paper we show that both $\mathscr{L}_O(d, i)$ and $\mathscr{L}_R(d, i)$ are atomic, and compute their characteristic polynomials.

1. Introduction

In this section we recall some terminologies and definitions about posets [1,2] and d-bounded distance-regular graphs.

Let P be a poset. For $a, b \in P$, we say *a covers b*, denoted by $b \lessdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. Let P be a finite poset with the minimum element, denoted by 0. By a *rank function* on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b \lessdot a$. Observe the rank function is unique if it exists. P is said to be *ranked* whenever P has a rank function. Let P be a finite poset with 0. By an *atom* we mean an element of P covering 0. We say P is *atomic* if each element in $P \setminus \{0\}$ is the *join* of atoms in P.

Let P be a ranked poset with 0 and the maximum element, denoted by 1. The polynomial

$$\chi(P, y) = \sum_{a \in P} \mu(0, a) y^{r(1)-r(a)}$$

is called the *characteristic polynomial* of P, where μ is the Möbius function on P and r is the rank function of P.

Let P and P' be two posets. If there exists a bijection σ from P to P' such that, for all $a, b \in P$, $a < b$ if and only if $\sigma(a) < \sigma(b)$, then σ is said to be an isomorphism from P to P'. In this case, we call P is isomorphic to P'. It is well known that two isomorphic posets have the same Möbius function.

Let $\Gamma = (X, R)$ be a connected regular graph. For vertices u and v of Γ, $\partial(u, v)$ denotes the *distance* between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $d(\Gamma)$. For vertices u and v at distance i, define

$$C(u, v) = C_i(u, v) = \{w \mid \partial(u, w) = i - 1, \partial(w, v) = 1\},$$
$$A(u, v) = A_i(u, v) = \{w \mid \partial(u, w) = i, \partial(w, v) = 1\}.$$

For the cardinalities of these sets we use lower case letters $c_i(u, v)$ and $a_i(u, v)$.

* Corresponding author.
E-mail address: wangks@bnu.edu.cn (K. Wang).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2009.08.014

A connected regular graph Γ with diameter d is said to be *distance-regular* if $c_i(u, v)$ and $a_i(u, v)$ depend only on i for all $1 \leq i \leq d$. The reader is referred to [3] for general theory of distance-regular graphs.

Recall that a subgraph induced on a subset Δ of X is said to be *strongly closed* if $C(u, v) \cup A(u, v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. Suzuki [18] determined all the types of strongly closed subgraphs of a distance-regular graph.

A distance-regular graph Γ with diameter d is said to be *d-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$.

Weng [23,24] used the term *weak-geodetically closed subgraphs* for strongly closed subgraphs, obtained the basic properties and characterized when a distance-regular graph is d-bounded. A regular strongly closed subgraph of Γ is said to be a *subspace* of Γ. For any two subspaces Δ_1 and Δ_2 of Γ, $\Delta_1 + \Delta_2$ denotes the minimum subspace containing Δ_1 and Δ_2.

The lattices generated by orbits of subspaces under finite classical groups were discussed in Huo, Liu and Wan [13–15], Huo and Wan [16], Gao and You [6,7], Wang and Feng [19], Wang and Guo [20,21], Guo and Nan [12,17], Guo [10], Guo, Li and Wang [11], Wang and Li [22]. The subspaces of a d-bounded distance-regular graph have similar properties to those of a vector space, as a generalization of the above results, some lattices were constructed by subspaces in a d-bounded distance-regular graph, see [4,8,9].

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 3$. For $x \in X$, let $P(x)$ be the set of all subspaces containing x in Γ. For $1 \leq i \leq d - 1$, let Δ_0 denote a fixed subspace with diameter $d - i$ in $P(x)$, and let

$$\mathscr{L}(d, i) = \{\Delta \in P(x) \mid \Delta + \Delta_0 = \Gamma, d(\Delta) = d(\Delta \cap \Delta_0) + i\} \cup \{\emptyset\}.$$

If we define the partial order on $\mathscr{L}(d, i)$ by ordinary inclusion (resp. reverse inclusion), then $\mathscr{L}(d, i)$ is a finite poset, denoted by $\mathscr{L}_O(d, i)$ (resp. $\mathscr{L}_R(d, i)$). In the present paper we show that both $\mathscr{L}_O(d, i)$ and $\mathscr{L}_R(d, i)$ are atomic, and compute their characteristic polynomials.

2. Some results on d-bounded distance-regular graphs

In this section we first recall some results on d-bounded distance-regular graphs, and then introduce two useful lemmas.

Proposition 2.1 (*[23, Lemma 2.6]*). *Let Γ be a d-bounded distance-regular graph. Then we have $b_i > b_{i+1}$ where $0 \leq i \leq d - 1$.*

Proposition 2.2 (*[24, Lemmas 4.2, 4.5]*). *Let Γ be a d-bounded distance-regular graph. Then the following hold:*

(i) *Let Δ be a subspace of Γ and $0 \leq i \leq d(\Delta)$. Then Δ is distance-regular with intersection numbers $c_i(\Delta) = c_i$, $a_i(\Delta) = a_i$, $b_i(\Delta) = b_i - b_{d(\Delta)}$.*
(ii) *For any vertices x and y, the subspace with diameter $\partial(x, y)$ containing x, y is unique.*

Proposition 2.3 (*[4, Lemma 2.1]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. For $1 \leq i + 1 \leq i + s \leq i + s + t \leq d$, suppose that Δ and Δ' are two subspaces satisfying $\Delta \subseteq \Delta'$, $d(\Delta) = i$ and $d(\Delta') = i + s + t$. Then the number of the subspaces with diameter $i + s$ containing Δ and contained in Δ', denoted by $N(i, i + s, i + s + t)$, is*

$$\frac{(b_i - b_{i+s+t})(b_{i+1} - b_{i+s+t}) \cdots (b_{i+s-1} - b_{i+s+t})}{(b_i - b_{i+s})(b_{i+1} - b_{i+s}) \cdots (b_{i+s-1} - b_{i+s})}.$$

Proposition 2.4 (*[4, Lemma 2.8]*). *Let Γ be a d-bounded distance-regular graph. Suppose that Δ and Δ' are two subspaces. If $d(\Delta \cap \Delta') \neq \emptyset$, then $d(\Delta) + d(\Delta') \geq d(\Delta \cap \Delta') + d(\Delta + \Delta')$.*

Proposition 2.5 (*[5, Lemma 2.8]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Let Δ and Δ' be two subspaces in Γ such that $d(\Delta) + d(\Delta') = d(\Delta \cap \Delta') + d(\Delta + \Delta')$. Then for all subspaces Δ_1 containing $\Delta \cap \Delta'$ in Δ, and for all subspaces Δ_2 containing $\Delta \cap \Delta'$ in Δ', we have $d(\Delta_1) + d(\Delta_2) = d(\Delta_1 \cap \Delta_2) + d(\Delta_1 + \Delta_2)$.*

Proposition 2.6 (*[5, Theorem 1.1]*). *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 3$. Suppose that $0 \leq t \leq i + t, j + t \leq i + j + t \leq d_1 \leq d$, and suppose that Δ and Δ^* are subspaces with diameter $i + t$ and diameter d_1 in $P(x)$, respectively. Suppose $\Delta \subseteq \Delta^*$. Then the number of subspaces Δ' with diameter $j+t$ and $\Delta' \subseteq \Delta^*$ in $P(x)$ such that $d(\Delta \cap \Delta') = t$ and $d(\Delta + \Delta') = i + j + t$, denoted by $M(t, i + t, j + t; d_1)$, is*

$$\frac{(b_0 - b_{i+t})(b_1 - b_{i+t}) \cdots (b_{t-1} - b_{i+t})(b_{i+t} - b_{d_1})(b_{i+t+1} - b_{d_1}) \cdots (b_{i+j+t-1} - b_{d_1})}{(b_0 - b_t)(b_1 - b_t) \cdots (b_{t-1} - b_t)(b_t - b_{j+t})(b_{t+1} - b_{j+t}) \cdots (b_{j+t-1} - b_{j+t})}.$$

In the rest of the paper, we always assume that $M(l, d - i, i + l; d)$ is given by Proposition 2.6. Now we prove two basic results on $\mathscr{L}(d, i)$.

Lemma 2.7. *Let $\Delta \in \mathscr{L}(d, i) \setminus \{\emptyset\}$ and $\Delta \supseteq \Delta_1 \in P(x)$. Then $\Delta_1 \in \mathscr{L}(d, i)$ if and only if $(\Delta \cap \Delta_0) + \Delta_1 = \Delta$ and $d(\Delta_1) = d(\Delta_1 \cap \Delta_0) + i$.*

Proof. Suppose $\Delta_1 \in \mathscr{L}(d, i)$. Then $\Delta_1 + \Delta_0 = \Gamma$ and $d(\Delta_1) = d(\Delta_1 \cap \Delta_0) + i$. Since $d(\Delta_1) + d(\Delta_0) = d(\Delta_1 \cap \Delta_0) + d(\Delta_1 + \Delta_0)$, by Proposition 2.5,

$$\begin{aligned} d((\Delta \cap \Delta_0) + \Delta_1) &= d(\Delta \cap \Delta_0) + d(\Delta_1) - d((\Delta \cap \Delta_0) \cap \Delta_1) \\ &= d(\Delta \cap \Delta_0) + d(\Delta_1) - d(\Delta_1 \cap \Delta_0) \\ &= d(\Delta \cap \Delta_0) + i \\ &= d(\Delta). \end{aligned}$$

By $\Delta_1 \subseteq \Delta$ we obtain $(\Delta \cap \Delta_0) + \Delta_1 \subseteq \Delta$, which implies that $\Delta = (\Delta \cap \Delta_0) + \Delta_1$.

Conversely, suppose $\Delta = (\Delta \cap \Delta_0) + \Delta_1$ and $d(\Delta_1) = d(\Delta_1 \cap \Delta_0) + i$. Then

$$\Delta_1 + \Delta_0 = \Delta_1 + ((\Delta \cap \Delta_0) + \Delta_0) = (\Delta_1 + (\Delta \cap \Delta_0)) + \Delta_0 = \Delta + \Delta_0 = \Gamma$$

implies that $\Delta_1 \in \mathscr{L}(d, i)$. □

Lemma 2.8. *Let $\Delta_1 \in \mathscr{L}(d, i) \setminus \{\emptyset\}$ and $\Delta_1 \subseteq \Delta$. Then $\Delta \in \mathscr{L}(d, i)$ if and only if $(\Delta \cap \Delta_0) + \Delta_1 = \Delta$.*

Proof. Since $d(\Delta_1) + d(\Delta_0) = d(\Delta_1 \cap \Delta_0) + d(\Delta_1 + \Delta_0)$, by Proposition 2.5,

$$d((\Delta \cap \Delta_0) + \Delta_1) = d(\Delta \cap \Delta_0) + d(\Delta_1) - d(\Delta_1 \cap \Delta_0) = d(\Delta \cap \Delta_0) + i.$$

Suppose $\Delta \in \mathscr{L}(d, i)$. Then $\Delta + \Delta_0 = \Gamma$ and $d(\Delta) = d(\Delta \cap \Delta_0) + i = d((\Delta \cap \Delta_0) + \Delta_1)$. Since $(\Delta \cap \Delta_0) + \Delta_1 \subseteq \Delta$, Proposition 2.2 implies $\Delta = (\Delta \cap \Delta_0) + \Delta_1$.

Conversely, suppose $\Delta = (\Delta \cap \Delta_0) + \Delta_1$. Then $\Delta + \Delta_0 \supseteq \Delta_1 + \Delta_0 = \Gamma$; and so $\Delta + \Delta_0 = \Gamma$. Since

$$d(\Delta) = d(\Delta_1 + (\Delta \cap \Delta_0)) = d(\Delta \cap \Delta_0) + i,$$

we obtain $\Delta \in \mathscr{L}(d, i)$. □

3. The poset $\mathscr{L}_O(d, i)$

In this section we prove that the poset $\mathscr{L}_O(d, i)$ is atomic, and compute its characteristic polynomial.

Theorem 3.1. *$\mathscr{L}_O(d, i)$ is atomic.*

Proof. Let $P(d, i; j)$ be the set of all subspaces Δ' with diameter j in $\mathscr{L}_O(d, i)$, where $i \leq j \leq d$. Since $\emptyset$ is the unique minimum element of $\mathscr{L}_O(d, i)$, $P(d, i; i)$ is the set of all atoms in $\mathscr{L}_O(d, i)$. In order to prove $\mathscr{L}_O(d, i)$ is atomic, it suffices to show that every element of $P(d, i; j)$ $(i \leq j \leq d)$ is the join of some atoms. The assertion is trivial for $j = i$. Suppose that the result is true for $j = i + l$. Let $\Delta \in P(d, i; i + l + 1)$. Then $d(\Delta \cap \Delta_0) = d(\Delta) - i = l + 1$. By Propositions 2.1 and 2.6, the number of subspaces $\Delta' \subseteq \Delta$ with diameter $i + l$ satisfying $(\Delta \cap \Delta_0) + \Delta' = \Delta$, $d(\Delta' \cap \Delta_0) = l$ and $\Delta' \in P(x)$ is

$$M(l, l+1, i+l; i+l+1) = \frac{(b_0 - b_{l+1}) \cdots (b_{l-1} - b_{l+1})(b_{l+1} - b_{i+l+1}) \cdots (b_{i+l} - b_{i+l+1})}{(b_0 - b_l) \cdots (b_{l-1} - b_l)(b_l - b_{i+l}) \cdots (b_{i+l-1} - b_{i+l})} \geq 2.$$

Lemma 2.7 implies that there exist two distinct subspaces $\Delta', \Delta'' \subseteq \Delta$ in $P(d, i; i + l)$. By Proposition 2.2 Δ is the join of Δ' and Δ''. By induction Δ is the join of some atoms. Therefore, $\mathscr{L}_O(d, i)$ is atomic. □

Lemma 3.2. *The Möbius function of $\mathscr{L}_O(d, i)$ is*

$$\mu(\Delta, \Delta') = \begin{cases} 1, & \text{if } \Delta = \Delta', \\ -1, & \text{if } \Delta \lessdot \Delta', \\ (-1)^s \dfrac{(b_{t+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_t - b_{t+1}) \cdots (b_t - b_{t+s-1})}, & \text{if } \emptyset \neq \Delta < \Delta', s \geq 2, \\ \displaystyle\sum_{j=0}^{t+s} (-1)^{t+s-j+1} \frac{(b_{j+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_j - b_{j+1}) \cdots (b_j - b_{t+s-1})} & \\ \quad \times M(j, t+s, i+j; i+s+t), & \text{if } \emptyset = \Delta < \Delta', s \geq i+1, \\ 0, & \text{otherwise}, \end{cases}$$

where $d(\Delta) = i + t$ and $d(\Delta') = i + t + s$ whenever $\Delta, \Delta' \neq \emptyset$, respectively.

Proof. Let $\emptyset \neq \Delta \subseteq \Delta'$. For any $\Delta \subseteq \tilde{\Delta} \in \mathscr{L}_O(d, i)$, we claim that $\tilde{\Delta} \subseteq \Delta'$ if and only if $\tilde{\Delta} \cap \Delta_0 \subseteq \Delta' \cap \Delta_0$. Suppose $\Delta \cap \Delta_0 \subseteq \tilde{\Delta} \cap \Delta_0 \subseteq \Delta' \cap \Delta_0$. Then by Lemma 2.8,

$$\tilde{\Delta} = (\tilde{\Delta} \cap \Delta_0) + \Delta, \Delta' = (\Delta' \cap \Delta_0) + \Delta.$$

It follows that $\Delta \subseteq \tilde{\Delta} \subseteq \Delta'$. The converse is obvious.

It is easy to verify that the mapping

$$\begin{aligned}\{\Delta_1 \mid \Delta \cap \Delta_0 \subseteq \Delta_1 \subseteq \Delta' \cap \Delta_0\} &\rightarrow \{\tilde{\Delta} \mid \Delta \subseteq \tilde{\Delta} \subseteq \Delta', \tilde{\Delta} \in \mathscr{L}_O(d, i)\}\\ \Delta_1 &\mapsto \Delta_1 + \Delta\end{aligned}$$

is an isomorphism between the two posets. Hence by Proposition 2.2 and [4, Theorem 4.1], the Möbius function of $\mathscr{L}_O(d, i)$ is

$$\mu(\Delta, \Delta') = \begin{cases} 1, & \text{if } \Delta = \Delta', \\ -1, & \text{if } \Delta \lessdot \Delta', \\ (-1)^s \dfrac{(b_{t+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_t - b_{t+1}) \cdots (b_t - b_{t+s-1})}, & \text{if } \emptyset \neq \Delta < \Delta', s \geq 2, \\ \displaystyle\sum_{\emptyset < \tilde{\Delta} \leq \Delta'} -\mu(\tilde{\Delta}, \Delta'), & \text{if } \emptyset = \Delta < \Delta', s \geq i+1, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2.7, the number of subspaces $\tilde{\Delta} \in \mathscr{L}_O(d, i)$ satisfying $\tilde{\Delta} \subseteq \Delta'$, $d(\tilde{\Delta}) = i+j\,(0 \leq j \leq t+s)$ is $M(j, t+s, i+j; i+s+t)$, which implies that

$$\sum_{\emptyset < \tilde{\Delta} \leq \Delta'} -\mu(\tilde{\Delta}, \Delta') = \sum_{j=0}^{t+s} (-1)^{t+s-j+1} M(j, t+s, i+j; i+s+t) \frac{(b_{j+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_j - b_{j+1}) \cdots (b_j - b_{t+s-1})},$$

as desired. $\square$

Theorem 3.3. *The characteristic polynomial of* $\mathscr{L}_O(d, i)$ *is*

$$\chi(\mathscr{L}_O(d, i), y) = y^{d-i+1} + \sum_{l=0}^{d-i} \sum_{j=0}^{l} (-1)^{l-j+1} M(l, d-i, i+l; d) M(j, l, i+j; i+l) \frac{(b_{j+1} - b_l) \cdots (b_{l-1} - b_l)}{(b_j - b_{j+1}) \cdots (b_j - b_{l-1})} y^{d-i-l}.$$

Proof. For any $\Delta \in \mathscr{L}_O(d, i)$, define

$$r_O(\Delta) = \begin{cases} 0, & \text{if } \Delta = \emptyset, \\ d(\Delta) - i + 1, & \text{otherwise.} \end{cases}$$

It is routine to check that r_O is the rank function on $\mathscr{L}_O(d, i)$.

By Proposition 2.6 and Lemma 3.2, we have

$$\begin{aligned}\chi(\mathscr{L}_O(d, i), y) &= \sum_{\Delta \in \mathscr{L}_O(d,i)} \mu(\emptyset, \Delta) y^{r_O(\Gamma) - r_O(\Delta)} \\ &= y^{d-i+1} + \sum_{\emptyset \neq \Delta \in \mathscr{L}_O(d,i)} \mu(\emptyset, \Delta) y^{d-d(\Delta)} \\ &= y^{d-i+1} + \sum_{l=0}^{d-i} \sum_{j=0}^{l} (-1)^{l-j+1} M(l, d-i, i+l; d) M(j, l, i+j; i+l) \frac{(b_{j+1} - b_l) \cdots (b_{l-1} - b_l)}{(b_j - b_{j+1}) \cdots (b_j - b_{l-1})} y^{d-i-l},\end{aligned}$$

as desired. $\square$

4. The poset $\mathscr{L}_R(d, i)$

In this section we prove that the poset $\mathscr{L}_R(d, i)$ is atomic, and compute its characteristic polynomial.

Theorem 4.1. $\mathscr{L}_R(d, i)$ *is atomic.*

Proof. Let $P(d, i; j)$ be the set of all subspaces Δ' with diameter j in $\mathscr{L}_R(d, i)$, where $i \leq j \leq d-1$. Since Γ is the unique minimum element, $P(d, i; d-1)$ is the set of all atoms in $\mathscr{L}_R(d, i)$. In order to prove $\mathscr{L}_R(d, i)$ is atomic, it suffices to show that every element of $P(d, i; j)$ $(i \le j \le d-1)$ is the join of some atoms. The assertion is trivial for $j = d-1$. Suppose that the result is true for $j = d-l$. Let $\Delta \in P(d, i; d-l-1)$. Then $d(\Delta \cap \Delta_0) = d(\Delta) - i = d-i-l-1$. By Propositions 2.1 and 2.3, the number of subspaces $\Delta' \subseteq \Delta_0$ with diameter $d-i-l$ containing $\Delta \cap \Delta_0$ in $P(x)$ is equal to

$$N(d-i-l-1, d-i-l, d-i) = \frac{b_{d-i-l-1} - b_{d-i}}{b_{d-i-l-1} - b_{d-i-l}} \ge 2.$$

Then there exist two distinct subspaces $\Delta', \Delta'' \subseteq \Delta_0$ with diameter $d-i-l$ in $P(x)$ containing $\Delta \cap \Delta_0$. Let $\Delta_1 = \Delta + \Delta'$, $\Delta_2 = \Delta + \Delta''$. Then $\Delta_1 + \Delta_0 = \Gamma$, $\Delta_2 + \Delta_0 = \Gamma$. Since $\Delta \cap \Delta' = \Delta \cap \Delta_0$, by Proposition 2.5

$$d(\Delta_1) = d(\Delta) + d(\Delta') - d(\Delta \cap \Delta') = d - l.$$

By Proposition 2.4,

$$d-i-l = d(\Delta') \le d(\Delta_1 \cap \Delta_0) \le d(\Delta_1) + d(\Delta_0) - d(\Delta_1 + \Delta_0) = d-i-l.$$

Since $\Delta' \subseteq \Delta_1 \cap \Delta_0$, $\Delta_1 \cap \Delta_0 = \Delta'$. Similarly, $\Delta_2 \cap \Delta_0 = \Delta''$. Lemma 2.8 implies that $\Delta_1, \Delta_2 \in P(d, i; d-l)$; and so Δ is the join of Δ_1 and Δ_2 by Proposition 2.2. By induction Δ is the join of some atoms. Therefore, $\mathscr{L}_R(d, i)$ is atomic. $\square$

Lemma 4.2. *The Möbius function of $\mathscr{L}_R(d, i)$ is*

$$\mu(\Delta, \Delta') = \begin{cases} 1, & \text{if } \Delta = \Delta', \\ -1, & \text{if } \Delta \lessdot \Delta', \\ (-1)^s \dfrac{(b_{t+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_t - b_{t+1}) \cdots (b_t - b_{t+s-1})}, & \text{if } \Delta < \Delta' \ne \emptyset, s \ge 2, \\ \displaystyle\sum_{j=0}^{t+s} (-1)^{t+s-j+1} \frac{(b_{j+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_j - b_{j+1}) \cdots (b_j - b_{t+s-1})} & \\ \quad \times M(j, t+s, i+j; i+s+t), & \text{if } \Delta < \Delta' = \emptyset, s \ge i+1, \\ 0, & \text{otherwise}, \end{cases}$$

where $d(\Delta) = i+t+s$ and $d(\Delta') = i+t$ whenever $\Delta, \Delta' \ne \emptyset$, respectively.

Proof. Similar to the proof of Lemma 3.2, by Proposition 2.2 and [4, Theorem 4.4], the Möbius function of $\mathscr{L}_R(d, i)$ is

$$\mu(\Delta, \Delta') = \begin{cases} 1, & \text{if } \Delta = \Delta', \\ -1, & \text{if } \Delta \lessdot \Delta', \\ (-1)^s \dfrac{(b_{t+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_t - b_{t+1}) \cdots (b_t - b_{t+s-1})}, & \text{if } \Delta < \Delta' \ne \emptyset, s \ge 2, \\ \displaystyle\sum_{\Delta \le \tilde{\Delta} < \emptyset} -\mu(\Delta, \tilde{\Delta}), & \text{if } \Delta < \Delta' = \emptyset, s \ge i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.7 implies that

$$\sum_{\Delta \le \tilde{\Delta} < \emptyset} -\mu(\Delta, \tilde{\Delta}) = \sum_{j=0}^{t+s} (-1)^{t+s-j+1} M(j, t+s, i+j; i+s+t) \frac{(b_{j+1} - b_{t+s}) \cdots (b_{t+s-1} - b_{t+s})}{(b_j - b_{j+1}) \cdots (b_j - b_{t+s-1})},$$

as desired. $\square$

Theorem 4.3. *The characteristic polynomial of $\mathscr{L}_R(d, i)$ is*

$$\chi(\mathscr{L}_R(d, i), y) = y^{d-i+1} + \sum_{l=0}^{d-i-1} (-1)^{d-i-l} \frac{(b_{l+1} - b_{d-i}) \cdots (b_{d-i-1} - b_{d-i})}{(b_l - b_{l+1}) \cdots (b_l - b_{d-i-1})} M(l, d-i, i+l; d) y^{l+1}$$
$$+ \sum_{j=0}^{d-i} (-1)^{d-i-j+1} M(j, d-i, i+j; d) \frac{(b_{j+1} - b_{d-i}) \cdots (b_{d-i-1} - b_{d-i})}{(b_j - b_{j+1}) \cdots (b_j - b_{d-i-1})}.$$

Proof. For any $\Delta \in \mathscr{L}_R(d, i)$, define

$$r_R(\Delta) = \begin{cases} d-i+1, & \text{if } \Delta = \emptyset, \\ d - d(\Delta), & \text{otherwise.} \end{cases}$$

It is routine to check that r_R is the rank function on $\mathscr{L}_R(d, i)$.

By Proposition 2.6 and Lemma 4.2, we have

$$\begin{aligned}\chi(\mathscr{L}_R(d,i),y) &= \sum_{\Delta\in\mathscr{L}_R(d,i)} \mu(\Gamma,\Delta)y^{r_R(\emptyset)-r_R(\Delta)}\\ &= y^{d-i+1} + \sum_{\Gamma\neq\Delta\in\mathscr{L}_R(d,i)} \mu(\Gamma,\Delta)y^{d(\Delta)-i+1}\\ &= y^{d-i+1} + \sum_{l=0}^{d-i-1}(-1)^{d-i-l}\frac{(b_{l+1}-b_{d-i})\cdots(b_{d-i-1}-b_{d-i})}{(b_l-b_{l+1})\cdots(b_l-b_{d-i-1})}M(l,d-i,i+l;d)y^{l+1}\\ &\quad + \sum_{j=0}^{d-i}(-1)^{d-i-j+1}M(j,d-i,i+j;d)\frac{(b_{j+1}-b_{d-i})\cdots(b_{d-i-1}-b_{d-i})}{(b_j-b_{j+1})\cdots(b_j-b_{d-i-1})},\end{aligned}$$

as desired. □

Acknowledgements

This research is supported by NCET-08-0052 and NSF of China (10871027). The authors are indebted to the referees who gave us many helpful suggestions.

References

[1] M. Aigner, Combinatorial Theory, Springer-Verlag, Berlin, 1979.
[2] G. Birgkhoff, Lattice Theory, third ed., American Mathematical Society, Providence, RI, 1967.
[3] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, New York, 1989.
[4] S. Gao, J. Guo, W. Liu, Lattices generated by strongly closed subgraphs in d-bounded distance-regular graphs, European J. Combin. 28 (2007) 1800–1813.
[5] S. Gao, J. Guo, B. Zhang, L. Fu, Subspaces in d-bounded distance-regular graphs and their applications, European J. Combin. 29 (2008) 592–600.
[6] Y. Gao, Lattices generated by orbits of subspaces under finite singular unitary group and its characteristic polynomials, Linear Algebra Appl. 368 (2003) 243–268.
[7] Y. Gao, H. You, Lattices generated by orbits of subspaces under finite singular classical groups and its characteristic polynomials, Comm. Algebra 31 (2003) 2927–2950.
[8] J. Guo, S. Gao, Lattices generated by join of strongly closed subgraphs in d-bounded distance-regular graphs, Discrete Math. 308 (2008) 1921–1929.
[9] J. Guo, S. Gao, K. Wang, Lattices generated by subspaces in d-bounded distance-regular graphs, Discrete Math. 308 (2008) 5260–5264.
[10] J. Guo, Lattices associated with finite vector spaces and finite affine spaces, Ars Combin. 88 (2008) 47–53.
[11] J. Guo, Z. Li, K. Wang, Lattices associated with totally isotropic subspaces in classical spaces, Linear Algebra Appl. 431 (2009) 1088–1095.
[12] J. Guo, J. Nan, Lattices generated by orbits of flats under finite affine-symplectic groups, Linear Algebra Appl. 431 (2009) 536–542.
[13] Y. Huo, Y. Liu, Z. Wan, Lattices generated by transitive sets of subspaces under finite classical groups I, Comm. Algebra 20 (1992) 1123–1144.
[14] Y. Huo, Y. Liu, Z. Wan, Lattices generated by transitive sets of subspaces under finite classical groups II, the orthogonal case of odd characteristic, Comm. Algebra 20 (1993) 2685–2727.
[15] Y. Huo, Y. Liu, Z. Wan, Lattices generated by transitive sets of subspaces under finite classical groups, the orthogonal case of even characteristic III, Comm. Algebra 21 (1993) 2351–2393.
[16] Y. Huo, Z. Wan, On the geomericity of lattices generated by orbits of subspaces under finite classical groups, J. Algebra 243 (2001) 339–359.
[17] J. Nan, J. Guo, Lattices generated by two orbits of subspaces under finite singular classical groups, Comm. Algebra (in press).
[18] H. Suzuki, On strongly closed subgraphs of highly regular graphs, European J. Combin. 16 (1995) 197–220.
[19] K. Wang, Y. Feng, Lattices generated by orbits of flats under finite affine groups, Comm. Algebra 34 (2006) 1691–1697.
[20] K. Wang, J. Guo, Lattices generated by orbits of totally isotropic flats under finite affine-classical groups, Finite Fields Appl. 14 (2008) 571–578.
[21] K. Wang, J. Guo, Lattices generated by two orbits of subspaces under finite classical groups, Finite Fields Appl. 15 (2009) 236–245.
[22] K. Wang, Z. Li, Lattices associated with vector space over a finite field, Linear Algebra Appl. 429 (2008) 439–446.
[23] C. Weng, D-bounded distance-regular graphs, European J. Combin. 18 (1997) 211–229.
[24] C. Weng, Classical distance-regular graphs of negative type, J. Combin. Theory Ser. B 76 (1999) 93–116.